Methods and applications of orthogonal matrix polynomials satisfying differential equations

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Seminar classical analysis
Leuven, June 11, 2008

Outline

1. Scalar versus matrix orthogonality
   - Scalar case
   - Matrix case
   - New phenomena
2. Applications
   - Quasi-birth-and-death processes
   - Quantum mechanics
   - Time-and-band limiting
3. Open problems

Outline

Scalar orthogonality

Let ω be a positive measure on $\mathbb{R}$. We can construct a family of orthonormal polynomials $(p_n)_n$, dense in $L^2(\omega, \mathbb{R})$ such that

$$\langle p_n, p_m \rangle = \int_{\mathbb{R}} p_n(t) p_m(t) d\omega(t) = \delta_{nm}, \quad n, m \geqslant 0$$

This is equivalent to a three term recurrence relation

$$tp_n(t) = a_{n+1} p_{n+1}(t) + b_n p_n(t) + a_n p_{n-1}(t), \quad a_{n+1} \neq 0, \quad b_n \in \mathbb{R} \quad n \geqslant 0$$

Jacobi operator (tridiagonal):

$$t \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ \vdots \end{pmatrix}$$

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Bochner problem

Bochner (1929): characterize $(p_n)_n$ satisfying

$$dp_n \equiv \underbrace{(\alpha_2 t^2 + \alpha_1 t + \alpha_0)}_{f_2(t)} p_n''(t) + \underbrace{(\beta_1 t + \beta_0)}_{f_1(t)} p_n'(t) = \lambda_n p_n(t)$$

This is equivalent to the symmetry of d with respect to $\langle \cdot, \cdot \rangle$,i.e.

$$\langle dp_n, p_m \rangle = \langle p_n, dp_m \rangle$$

Moment equations

$$(n-1)(\alpha_2\mu_n+\alpha_1\mu_{n-1}+\alpha_0\mu_{n-2})+\beta_1\mu_n+\beta_0\mu_{n-1}, \; n \geqslant 1, \; \mu_n = \int t^n \omega(t)dt$$

Pearson equation

$$(f_2(t)\omega(t))' = f_1(t)\omega(t)$$

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Classical families

Hermite: $f_2(t) = 1, \omega(t) = e^{-t^2}$, $t \in (-\infty, \infty)$:

$$H_n(t)'' - 2tH_n(t)' = -2nH_n(t)$$

Laguerre: $f_2(t) = t, \omega(t) = t^{\alpha} e^{-t}$, $\alpha > -1$, $t \in (0, \infty)$:

$$tL_n^{\alpha}(t)'' + (\alpha + 1 - t)L_n^{\alpha}(t)' = -nL_n^{\alpha}(t)$$

Jacobi: $f_2(t) = t(1-t), \omega(t) = t^{\alpha}(1-t)^{\beta}$, $\alpha, \beta > -1$, $t \in (0,1)$:

$$t(1-t)P_n^{(\alpha,\beta)}(t)'' + (\alpha + 1 - (\alpha+\beta+2)t)P_n^{(\alpha,\beta)}(t)' =$$

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Applications:

- Quantum non relativistic models (Schrödinger equation).
- Electrostatic equilibrium (with logarithmic potential).

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Matrix case

Matrix valued polynomials on the real line:

$$C_n t^n + C_{n-1} t^{n-1} + \cdots + C_0, \quad C_i \in \mathbb{C}^{N\times N}$$

Krein (1949): orthogonal matrix polynomials (OMP)
Orthogonality: weight matrix W (positive definite on $L^2(W, \mathbb{C}^{N\times N})$)
Matrix valued inner product:

$$\langle P, Q\rangle_W = \int_a^b P(t)\mathrm{d}W(t)Q^*(t) \in \mathbb{C}^{N\times N}, \quad P, Q \in \mathbb{C}^{N\times N}[t]$$

- A weight matrix $W(t)$ reduces to scalar weights if there exists a nonsingular matrix (independent of t) T such that $W(t) = TD(t)T^*$ where $D(t)$ diagonal.

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Orthonormality of $(P_n)_n$ with respect to a weight matrix W

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- Systematic study: Asymptotics, zeros of OMP, quadrature formulae...
  Applications: scattering theory, times series and signal processing...

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$n \geqslant 0, \quad \Lambda_n \quad$ Hermitian

Equivalent to the symmetry of

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^1 F_0(t), \quad \partial = \frac{\mathrm{d}}{\mathrm{d}t}$$

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How to get examples

- Matrix spherical functions associated to $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$ Grünbaum-Pacharoni-Tirao (2003)
- Durán-Grünbaum (2004):

Moment equations

$$B_n^2 = (B_n^2)^*, \quad n \geqslant 2$$

$$2(n-1)B_n^2 + B_n^1 + (B_n^1)^* = 0, \quad n \geqslant 1$$

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$$B_n^j = \sum_{i=0}^{j} F_{j-i}^j \mu_{n-i}, \quad j = 0, 1, 2, \quad n \geqslant j, \quad \mu_n = \int t^n dW(t)$$

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$$\lim_{t\to x} F_2(t)W(t) = 0 = \lim_{t\to x} (F_1(t)W(t) - W(t)F_1^*(t)), \quad \text{for } x = a, b$$

General method: Suppose $F_2(t) = f_2(t)$ with real coefficients. Factorize

$$W(t) = \omega(t)T(t)T^*(t),$$

where ω is an scalar weight (Hermite, Laguerre or Jacobi) and T is a matrix function solving

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1. The first symmetry equation is trivial.
2. Defining
$$F_1(t) = 2f_2(t)G(t) + \frac{(f_2(t)\omega(t))'}{\omega(t)}$$
the second symmetry equation also holds.
3. The third is equivalent to
$$(F_1W - WF_1^*)' = 2(F_0W - WF_0^*).$$
Then, it is enough to find F_0 such that
$$\chi(t) = T^{-1}(t)\left(f_2(t)G(t) + f_2(t)G(t)^2 + \frac{(f_2(t)\omega(t))'}{\omega(t)}G(t) - F_0\right)T(t)$$
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Examples

Take $f_2 = I$ and $\omega = e^{-t^2}$.

Then $G(t) = A + 2Bt$

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The Hermitian condition force us to take

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \nu_i \in \mathbb{C} \setminus \{0\}$$

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and for $f_2 = (1-t^2)I$ and $\omega = (1-t)^{\alpha}(1+t)^{\beta}$
Then $G(t) = \frac{A}{1-t} + \frac{B}{1+t}$

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- Matrix valued bispectral problem (Grünbaum-Tirao, 2007) $\Rightarrow$ *ad-conditions*

$$\left\{\begin{array}{l} \underbrace{\begin{pmatrix} B_0 & A_1 & & \\ A_1^* & B_1 & A_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}}_{\mathcal{L}} \begin{pmatrix} P_0(t) \\ P_1(t) \\ \vdots \end{pmatrix} = t \begin{pmatrix} P_0(t) \\ P_1(t) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} P_0(t) \\ P_1(t) \\ \vdots \end{pmatrix} D = \underbrace{\begin{pmatrix} \Lambda_0 & & \\ & \Lambda_1 & \\ & & \ddots \end{pmatrix}}_{\Lambda} \begin{pmatrix} P_0(t) \\ P_1(t) \\ \vdots \end{pmatrix} \end{array}\right. \Leftrightarrow \mathrm{ad}_{\mathcal{L}}^{k+1}(\Lambda) = 0$$

where $\mathcal{L}$ is the Jacobi operator of the corresponding family of OMP and D is a differential operator of order k

Algebra of differential operators

For a fixed family $(P_n)_n$ of OMP we study the algebra over $\mathbb{C}$

$$\mathcal{D}(W) = \left\{ D = \sum_{i=0}^{k} \partial^i F_i(t) : P_n D = \Lambda_n(D) P_n, \ n = 0, 1, 2, \ldots \right\}$$

Scalar case: If $\mathcal{F}$ is the second order differential operator (Hermite, Laguerre or Jacobi), then any operator $\mathcal{U}$ such that $\mathcal{U} p_n = \lambda_n p_n$

$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{F}^i, \quad c_i \in \mathbb{C}$$

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Matrix case: This algebra can be noncommutative and generated by several elements

- Existence of several linearly independent second order differential operators having a fixed family of MOP as eigenfunctions
- Existence of families of MOP satisfying odd order differential equations

Algebras: conjectures (Castro, Durán, Grünbaum, MdI) except one (Tirao) due to Castro–Grünbaum (2006)
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- The map $D \mapsto (\Lambda_n(D))_n$ is a *faithful representation*, i.e.
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Convex cone of weight matrices

Dual situation to $\mathcal{D}(W)$: given a fixed differential operator D we study:

$$\Upsilon(D) = \{W : \langle PD, Q\rangle_W = \langle P, QD\rangle_W, \quad \text{for all} \quad P, Q\}$$

- If $\Upsilon(D) \neq \emptyset$, it is a convex cone:
  $W_1, W_2 \in \Upsilon(D) \Rightarrow \gamma W_1 + \zeta W_2 \in \Upsilon(D)$, $\gamma, \zeta \geqslant 0$ (one of them $\neq 0$)

The weight matrices W going along with a symmetric second order differential operator D give examples where $\Upsilon(D) \neq \emptyset$ (one dimensional)

We show the first examples of symmetric second order differential operators D for which $\Upsilon(D)$ is a two dimensional convex cone.

$\Rightarrow$ New phenomenon: (Monic) MOP $P_{n,\zeta/\gamma}$ with respect to $\gamma W_1 + \zeta W_2$

$$P_{n,\zeta/\gamma} D = \Gamma_n P_{n,\zeta/\gamma}$$

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Adding a Dirac delta distribution

All examples we consider are of the form

$$\gamma W + \zeta M(t_0)\delta_{t_0}, \quad \gamma > 0, \zeta \geqslant 0, \quad t_0 \in \mathbb{R},$$

where W is a weight matrix having several linearly independent symmetric second order differential operators and $M(t_0)$ certain positive semidefinite matrix.

Scalar case ($\omega + m\delta_{t_0}$)

- Second order: there are NOT symmetric second order differential operators.
- Fourth order: t_0 at the endpoints of the support, which is NOT symmetric with respect to the original weight (Krall, 1941):

$$\text{Laguerre type } e^{-t} + M\delta_0$$
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Method to find examples

Theorem (Durán–MdI, 2008)

Let W be a weight matrix and $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$. Assume that associated with the real point $t_0 \in \mathbb{R}$ there exists a Hermitian positive semidefinite matrix $M(t_0)$ satisfying

$$\begin{aligned} F_2(t_0)M(t_0) &= 0, \\ F_1(t_0)M(t_0) &= 0, \\ F_0 M(t_0) &= M(t_0)F_0^* \end{aligned}$$

Then

$$D \text{ is symmetric with respect to } W$$
$$\Leftrightarrow$$
$$D \text{ is symmetric with respect to } \gamma W + \zeta M(t_0)\delta_{t_0}$$

Example where $t_0 \in \mathbb{R}$

$$W(t) = e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}$$

Symmetry equations $\Rightarrow$ Expression for the 5-dimensional (real) linear space of symmetric differential operators of order at most two

Constraints:

$$F_2(t_0) M(t_0) = 0,$$
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$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t),$$

$$F_2(t) = \begin{pmatrix} 1 - at & -1 + a^2t^2 \\ -1 & 1 + at \end{pmatrix}$$

$$F_1(t) = \begin{pmatrix} -2a - 2t & 2a + 2(2 + a^2)t \\ 0 & -2t \end{pmatrix}$$

$$F_0(t) = \begin{pmatrix} -1 & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ D is symmetric with respect to the family of weight matrices

$$\Upsilon(D) = \left\{ \gamma e^{-t^2} \begin{pmatrix} 1 + a^2t^2 & at \\ at & 1 \end{pmatrix} + \zeta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_0(t), \quad \gamma > 0, \zeta \geqslant 0 \right\}$$

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$$F_2(t) = \begin{pmatrix} -\xi^{\mp}_{a,t_0} + at_0 - at & -1 - (a^2 t_0)t + a^2 t^2 \\ -1 & -\xi^{\mp}_{a,t_0} + at \end{pmatrix}$$

$$F_1(t) = \begin{pmatrix} -2a + 2\xi^{\mp}_{a,t_0} t & -2t_0 - 2a\xi^{\mp}_{a,t_0} + 2(2+a^2)t \\ 2t_0 & 2(\xi^{\mp}_{a,t_0} - at_0)t \end{pmatrix}$$

$$F_0(t) = \begin{pmatrix} \xi^{\mp}_{a,t_0} + 2\frac{t_0}{a} & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & -\xi^{\mp}_{a,t_0} - 2\frac{t_0}{a} \end{pmatrix}$$

$$M(t_0) = \begin{pmatrix} (\xi^{\pm}_{t_0,a})^2 & \xi^{\pm}_{t_0,a} \\ \xi^{\pm}_{t_0,a} & 1 \end{pmatrix}, \quad \xi^{\pm}_{a,t_0} = \frac{at_0 \pm \sqrt{4 + a^2 t_0^2}}{2}$$

Outline

Birth-and-death processes

Transition probability matrix

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_n \geqslant 0, a_n, c_n > 0, \quad a_n + b_n + c_n = 1$$

Birth-and-death processes

Transition probability matrix

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_n \geqslant 0, a_n, c_n > 0, \quad a_n + b_n + c_n = 1$$

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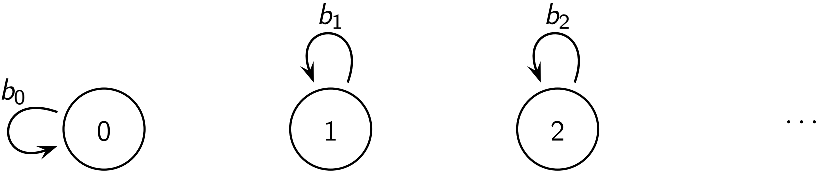

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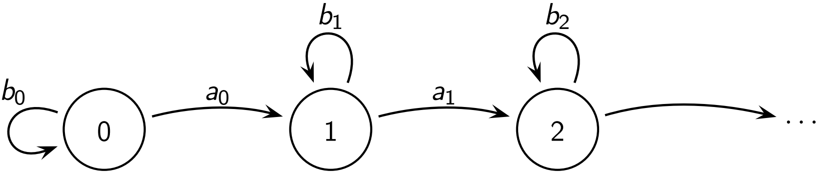

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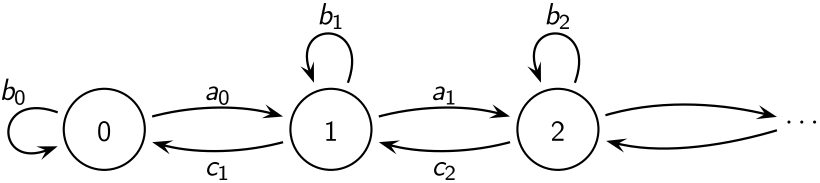

Introducing the polynomials $(q_n)_n$ by the conditions $q_{-1}(t) = 0$, $q_0(t) = 1$ and the recursion relation

$$t \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \end{pmatrix} = P \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \end{pmatrix}$$

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$$tq_n(t) = a_n q_{n+1}(t) + b_n q_n(t) + c_n q_{n-1}(t), \quad n = 0, 1, \ldots$$

there exists a unique measure $\mathrm{d}\omega(t)$ supported in $[-1, 1]$ such that

$$\int_{-1}^{1} q_i(t) q_j(t) \mathrm{d}\omega(t) \Big/ \int_{-1}^{1} q_j(t)^2 \mathrm{d}\omega(t) = \delta_{ij}$$

n-step transition probability matrix:

$$\mathrm{Prob}\{E_i \to E_j \text{ in } n \text{ steps}\} = P_{ij}^n = \sum_{k_1,k_2,\dots,k_{n-1}} P_{ik_1} P_{k_1 k_2} \cdots P_{k_{n-1} j}$$

Karlin y McGregor (1959): integral representation of P^n

Karlin-McGregor formula

$$P_{ij}^n = \int_{-1}^{1} t^n q_i(t) q_j(t) d\omega(t) \Big/ \int_{-1}^{1} q_j(t)^2 d\omega(t)$$

Invariant measure or distribution

A non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \dots)$ with non-negative components

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

$$\Rightarrow \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\int_{-1}^{1} q_i^2(t) d\omega(t)} = \frac{1}{\|q_i\|^2}$$

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Quasi-birth-and-death processes

Transition probability matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \begin{array}{l} (A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geqslant 0, \ \det(A_n), \det(C_n) \neq 0 \\ \displaystyle\sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1, \ i = 1, \ldots, N \end{array}$$

Particular case: pentadiagonal matrix

$$P = \left(\begin{array}{cc|cc|cc|cc|c} b_0 & a_0 & d_0 & 0 & & 0 & & & \\ c_1 & b_1 & a_1 & d_1 & & & & & \\ \hline e_2 & c_2 & b_2 & a_2 & d_2 & 0 & & 0 & \\ 0 & e_3 & c_3 & b_3 & a_3 & d_3 & & & \\ \hline & 0 & e_4 & c_4 & b_4 & a_4 & d_4 & 0 & \ddots \\ & & 0 & e_5 & c_5 & b_5 & a_5 & d_5 & \\ \hline & & & \ddots & & \ddots & & \ddots & \ddots \end{array}\right)$$

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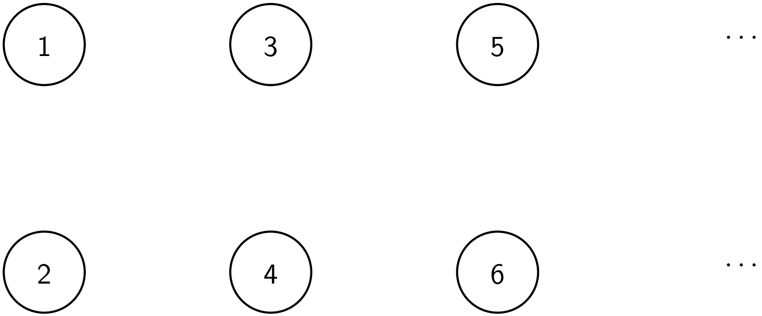
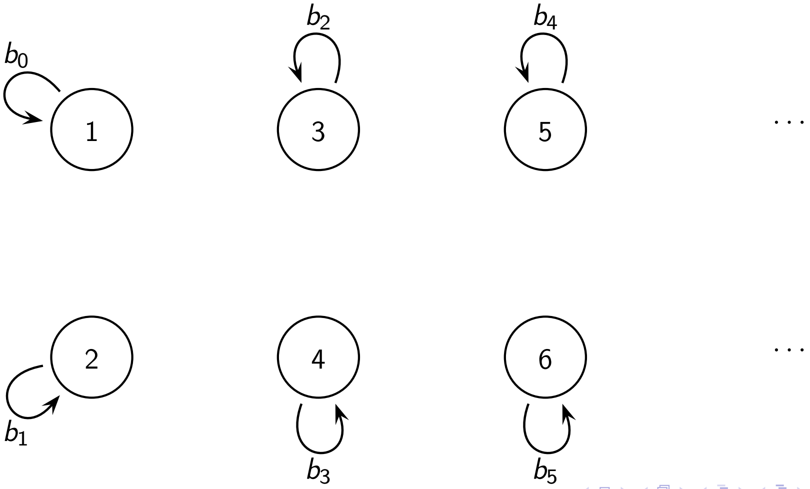
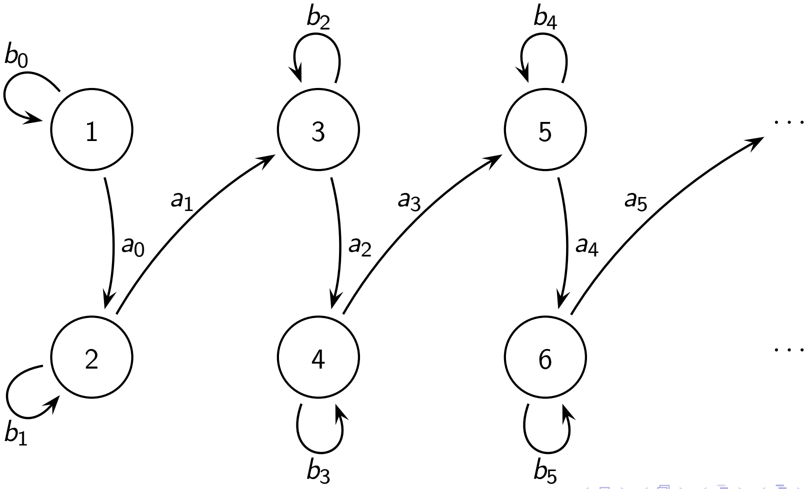
Network



Network



Network



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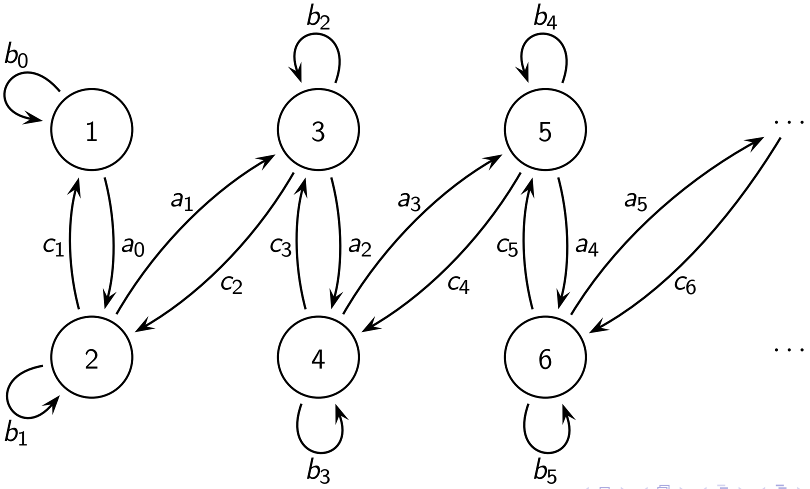

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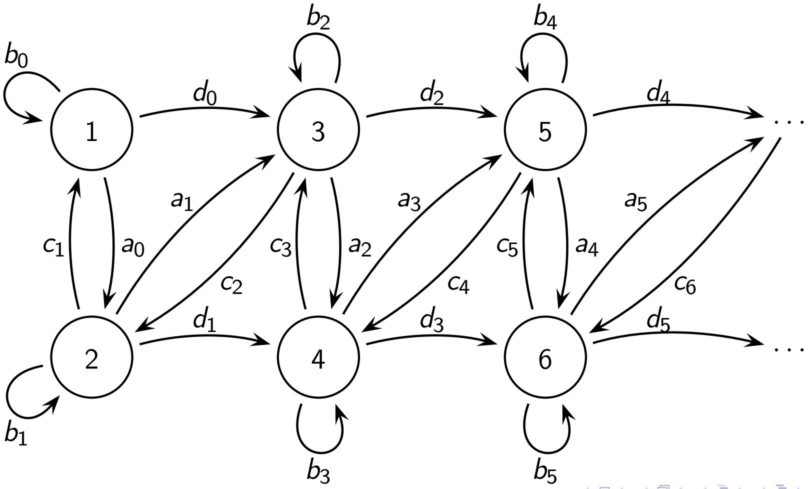
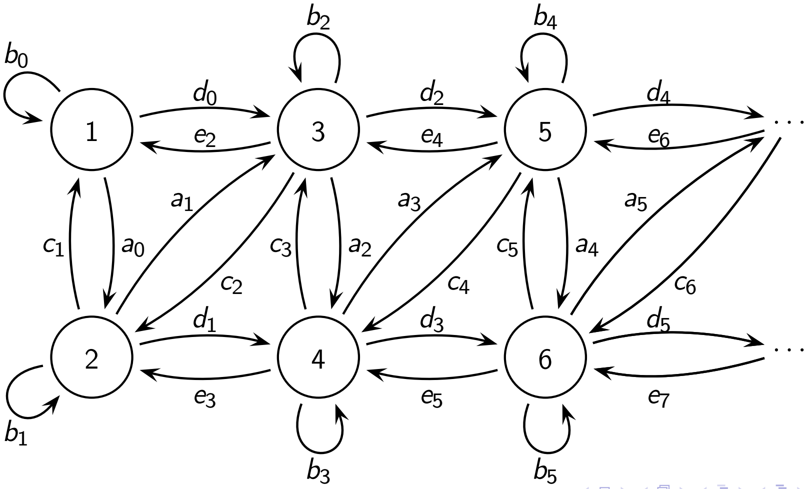

b_0
b_1
b_2
b_3
b_4
b_5
1
2
3
4
5
6
d_0
d_1
d_2
d_3
d_4
d_5
a_0
a_1
a_2
a_3
a_4
a_5
c_1
c_2
c_3
c_4
c_5
c_6
...
...

Network


b_0
b_1
b_2
b_3
b_4
b_5
1
2
3
4
5
6
d_0
d_1
d_2
d_3
d_4
d_5
e_2
e_3
e_4
e_5
e_6
e_7
a_0
a_1
a_2
a_3
a_4
a_5
c_1
c_2
c_3
c_4
c_5
c_6
$\cdots$
$\cdots$

OMP: Grünbaum (2007) and Dette-Reuther-Studden-Zygmunt (2007):
Introducing the matrix polynomials $(Q_n)_n$ by the conditions $Q_{-1}(t)=0$, $Q_0(t)=I$ and the recursion relation

$$t\begin{pmatrix}Q_0(t)\\ Q_1(t)\\ \vdots\end{pmatrix}=P\begin{pmatrix}Q_0(t)\\ Q_1(t)\\ \vdots\end{pmatrix}$$

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$$tQ_n(t)=A_nQ_{n+1}(t)+B_nQ_n(t)+C_nQ_{n-1}(t),\quad n=0,1,\ldots$$

and under certain technical conditions over A_n, B_n, C_n, there exists an unique weight matrix $dW(t)$ supported in $[-1,1]$ such that

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Karlin-McGregor formula

$$P_{ij}^n = \left(\int_{-1}^{1} t^n Q_i(t) \mathrm{d}W(t) Q_j^*(t)\right)\left(\int_{-1}^{1} Q_j(t) \mathrm{d}W(t) Q_j^*(t)\right)^{-1}$$

Invariant measure or distribution

Non-null vector with non-negative components

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots) \equiv (\pi_1^0, \pi_2^0, \ldots, \pi_N^0; \pi_1^1, \pi_2^1, \ldots, \pi_N^1; \cdots)$$

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The family of processes (size $N \times N$)

Conjugation

$$W(t) = T^* \widetilde{W}(t) T$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & -\dfrac{\alpha+\beta-k+2}{\beta-k+1} \end{pmatrix}$$

Grünbaum-MdI (2008)

$$\widetilde{W}(t) = t^{\alpha}(1-t)^{\beta} \begin{pmatrix} kt+\beta-k+1 & (1-t)(\beta-k+1) \\ (1-t)(\beta-k+1) & (1-t)^2(\beta-k+1) \end{pmatrix}$$

$t \in (0,1)$, $\alpha, \beta > -1$, $0 < k < \beta+1$

Pacharoni-Tirao (2006)

We consider the family of OMP $(Q_n(t))_n$ such that

- Three term recurrence relation

$$tQ_n(t) = A_nQ_{n+1}(t) + B_nQ_n(t) + C_nQ_{n-1}(t), \quad n = 0, 1, \dots$$

where the Jacobi matrix is stochastic

- Choosing $Q_0(t) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)}\begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)}\times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

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Pentadiagonal Jacobi matrix

Particular case $\alpha = \beta = 0$, $k = 1/2$:

$$P = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{50} & & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} & \\ & & & & \frac{81}{392} & \frac{3}{196} & \frac{1955}{3969} & \frac{5}{324} & \frac{175}{648} \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Invariant measure

Invariant measure

The row vector

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots)$$

$$\boldsymbol{\pi}^n = \left(\frac{1}{(\|Q_n\|_W^2)_{1,1}}, \frac{1}{(\|Q_n\|_W^2)_{2,2}}, \cdots, \frac{1}{(\|Q_n\|_W^2)_{N,N}} \right), \quad n \geqslant 0$$

is an invariant measure of P

Particular case $N = 2$, $\alpha = \beta = 0$, $k = 1/2$:

$$\pi^n = \left(\frac{2(n+1)^3}{(2n+3)(2n+1)}, \frac{(n+1)(n+2)}{2n+3} \right), \quad n \geqslant 0$$

$$\pi = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \cdots \right)$$

Invariant measure

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The row vector

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Quantum mechanics

Dirac's equation (central Coulomb potential)

$$T'(t) = \left(A + \frac{B}{t}\right) T(t)$$

where

$$A = \begin{pmatrix} 0 & 1+\omega \\ 1-\omega & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -a & b \\ -b & a \end{pmatrix}$$

Rose (1961)

Choosing $\omega = \pm\sqrt{a^2 - b^2}/a$ (lowest possible energy level) the solution of the Dirac's equation gives rise to a matrix weight whose OMP are eigenfunctions of certain second order differential equation

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Theorem (Durán-Grünbaum, 2006)

Consider the following instance of the Dirac's equation

$$T'(t) = \left(\widetilde{A} + \frac{\widetilde{B}}{t}\right) T(t)$$

$$\widetilde{A} = \sqrt{1 - 1/(4a)^2} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} -1/2 & a - 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Then $W(t) = t^{\alpha+1} e^{-t} T(t) e^{-D_{\widetilde{A}} t} H e^{-D^*_{\widetilde{A}} t} T^*(t)$, *where* $H = e^{D_{\widetilde{A}}} T^{-1}(1) (T^{-1})^*(1) e^{D^*_{\widetilde{A}}}$ *allows for the following second order differential operator*

$$D = \partial^2 tI + \partial^1 (-tI + 2E + (\alpha + 1)I) + \partial^0 (-E + E_0)$$

$$E = \begin{pmatrix} 0 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad E_0 = \frac{1 + \alpha}{5} \begin{pmatrix} -1 & 1/2 \\ -2 & 1 \end{pmatrix}$$

Time-and-band limiting

Given a full matrix M (integral operator) the computation of all its eigenvectors can be explicitly given if one finds a tridiagonal matrix S (differential operator) with simple spectrum such that

$$MS = SM$$

Classical scalar orthogonal polynomials: Grünbaum (1983)

Matrix case: Durán-Grünbaum (2005)

Example of QBD for $N = 2, \alpha = \beta = 0, k = 1/2$

$$W(t) = \begin{pmatrix} \frac{1}{2}t + \frac{1}{2} & 2t - 1 \\ 2t - 1 & \frac{9}{2}t^2 - \frac{11}{2}t + 2 \end{pmatrix}, \quad t \in [0, 1]$$

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Considering the same family $(Q_n)_n$ as before we have that

$$\|Q_n\|_W^2 = \begin{pmatrix} \frac{(2n+1)(2n+3)}{2(n+1)^3} & 0 \\ 0 & \frac{2n+3}{(n+1)(n+2)} \end{pmatrix}$$

and we can produce a family of normalized OMP $P_n = \|Q_n\|_W^{-1} Q_n$

Reproducing kernel

$$(M)_{i,j} = \int_0^\Omega P_i(t)W(t)P_j^*(t)\mathrm{d}t, \quad i,j = 0,1,\ldots,T$$

"Band limiting": Restriction to the interval $(0,\Omega)$

"Time limiting": Restriction to the range $0,1,\ldots,T$

$\Rightarrow$ There exists a block tridiagonal matrix S (pentadiagonal) such that M commutes with S

Scalar case: the vector space of all possible S's is 2-dimensional

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Outline

Open problems

- Classify all families of OMP satisfying any order differential operators.
- Proof of the algebras of differential operators associated with any size weight matrix.
- Electrostatic equilibrium of the zeros of these new families of OMP.
- Riemann-Hilbert problem for MOP: Given a weight matrix W and a positive integer n, find a $2N \times 2N$ matrix valued function
  1. $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2N \times 2N}$ is analytic.
  2. Y has boundary values for $t \in \mathbb{R}$, denoted by $Y_{\pm}(t)$, and
  $$Y_+(t) = Y_-(t) \begin{pmatrix} I & W(t) \\ 0 & I \end{pmatrix}, \quad t \in \mathbb{R}$$
  3. As $z \to \infty$,
  $$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n I & 0 \\ 0 & z^{-n} I \end{pmatrix}$$

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The unique solution of the Riemann-Hilbert problem is given by

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$P_n(z)$ monic MOP,γ_n is the leading coefficient of an orthonormal family.

- Asymptotics of these new families of OMP: Try to find the matrix version of the Heine's formula

$$P_n(x) = \frac{1}{n!D_n}\int\cdots\int\prod_{j=1}^{n}(x-x_j)\prod_{i<j}(x_j-x_i)^2 d\omega(x_1)\cdots d\omega(x_n)$$

where

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}$$

using quasi-determinants.

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